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2007 J. Phys. A: Math. Theor. 40 F491

(http://iopscience.iop.org/1751-8121/40/24/F08)

IOP PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F491–F501 doi:10.1088/1751-8113/40/24/F08

FAST TRACK COMMUNICATION

Isomonodromic deformation theory and the next-to-diagonal correlations of the anisotropic square lattice Ising model

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Received 20 March 2007, in final form 3 May 2007
Published 30 May 2007
Online at stacks.iop.org/JPhysA/40/F491

Abstract

In 1980, Jimbo and Miwa evaluated the diagonal two-point correlation function of the square lattice Ising model as a τ-function of the sixth Painlevé system by constructing an associated isomonodromic system within their theory of holonomic quantum fields. More recently an alternative isomonodromy theory was constructed based on bi-orthogonal polynomials on the unit circle with regular semi-classical weights, for which the diagonal Ising correlations arise as the leading coefficient of the polynomials specialized appropriately. Here we demonstrate that the next-to-diagonal correlations of the anisotropic Ising model are evaluated as one of the elements of this isomonodromic system or essentially as the Cauchy–Hilbert transform of one of the bi-orthogonal polynomials.

PACS numbers: 02.30.Ik, 02.30.Hq, 02.30.Gp, 02.30.−f, 05.50.+q, 75.10.Hk
Mathematics Subject Classification: 82B20, 34M55, 33C45

For the square lattice Ising model on the infinite lattice, an unpublished result of Onsager (see [13]) gives that the diagonal spin–spin correlation $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ has the Toeplitz determinant form

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = \det(a_{i-j}(k))_{1\leqslant i,j\leqslant N}, \tag{1}$$

where the elements are given by

$$a_n = \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\frac{k\cos n\theta - \cos(n-1)\theta}{\sqrt{k^2+1-2k\cos\theta}}. \tag{2}$$

A significant development occurred when Jimbo and Miwa [10, 11] identified (1) as the τ-function of a $\mathrm{P_{VI}}$ system. This identification has the consequence of allowing (1) to be characterized in terms of a solution of the σ-form of the Painlevé VI equation, a second-order second-degree ordinary differential equation with respect to $t := k^{\pm 2}$ with parameter N or as the solution of coupled recurrence relations in N with parameter t, which were subsequently shown to be equivalent to the discrete Painlevé V equation. This was derived from the monodromy preserving deformation of a certain linear system as a particular example of their general theory of holonomic quantum fields [14, 17, 12]; however, the theoretical machinery employed there was never put to use on related problems arising from the Ising model. See the forthcoming monograph [16] on recent progress utilizing this viewpoint. In a recent work [6] Forrester and the present author identified (1) as a τ-function in the Okamoto theory of P_{VI} [15], and subsequently developed an alternative isomonodromic theory [8] founded on bi-orthogonal systems on the unit circle with regular semi-classical weights. We remark that the result of Borodin [4] can also be used for the same purpose.

In a further development, Au-Yang and Perk [2, 1] discovered that the next-to-diagonal spin–spin correlations have the bordered Toeplitz determinant form

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \det \begin{pmatrix} a_0 & \cdots & a_{-N+2} & b_{N-1} \\ a_1 & \cdots & a_{-N+3} & b_{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N-1} & \cdots & a_1 & b_0 \end{pmatrix}, \qquad N \geqslant 1, \tag{3}$$

where the elements a_n are the same as those above and b_n are given by

$$b_n = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{\bar{C}}{\sqrt{k^2+1-2k\cos\theta}} \frac{(k\bar{S}-S)\cos n\theta + kS\cos(n-1)\theta - \bar{S}\cos(n+1)\theta}{S^2+\bar{S}^2+2k\cos\theta} \tag{4}$$

(the definitions of the model parameters $k, S, \bar{S}$ are given in the following paragraph). The task of the present study is to answer the following questions: can this correlation be evaluated in terms of a Painlevé-type function and if so which one? The answer is in the affirmative, and we identify the function in proposition 3. In order to understand the result for the next-to-diagonal correlations in its proper context, we will need to revise some relevant known results for the diagonal correlations. In fact even in an algorithmic sense in order to compute the next-to-diagonal correlations, one has to first compute the diagonal ones.

Consider the Ising model with spins $\sigma_r \in \{-1, 1\}$ located at site $r = (i, j)$ on a square lattice of dimension $(2L+1)\times(2L+1)$, centred about the origin. The first co-ordinate of a site refers to the horizontal or x-direction and the second to the vertical or y-direction (see figure 1), which is the convention opposite to that of McCoy and Wu [13] and early studies where the first co-ordinate labelled the rows in an ascending order and the second the columns from left to right. We will focus on the homogeneous but anisotropic Ising model, where the dimensionless nearest neighbour couplings are equal to $\bar{K}$ and K in the x- and y-directions respectively (see e.g. [3] and figure 1).

The probability density function for configuration $\{\sigma_{ij}\}_{i,j=-L}^{L}$ is given by

$$\Pr\left(\{\sigma_{ij}\}_{i,j=-L}^{L}\right) = \frac{1}{Z_{2L+1}} \exp\left[\bar{K}\sum_{j=-L}^{L}\sum_{i=-L}^{L-1}\sigma_{ij}\sigma_{i+1j} + K\sum_{i=-L}^{L}\sum_{j=-L}^{L-1}\sigma_{ij}\sigma_{ij+1}\right], \tag{5}$$

and averages are defined by

$$\langle\cdot\rangle = \sum_{\sigma_{ij}} \Pr\left(\{\sigma_{ij}\}_{i,j=-L}^{L}\right). \tag{6}$$

The normalization Z_{2L+1} is the partition function and conventionally periodic boundary conditions, $\sigma_{i,L+1} = \sigma_{i,-L}, \sigma_{L+1,j} = \sigma_{-L,j}$ for all i, j, are taken for convenience. In all

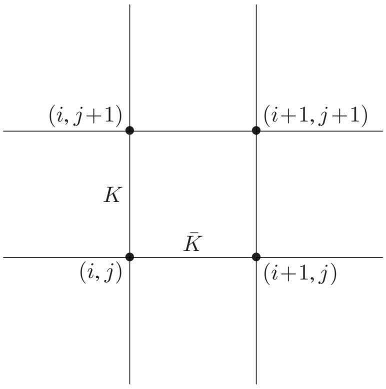

Figure 1. Co-ordinate system and couplings for the homogeneous anisotropic square lattice Ising model.

such averages, the thermodynamic limit is taken as $\lim_{L\to\infty}\langle\cdot\rangle$ keeping $K, \bar{K}$ fixed. The relevant variables in our study are the following variables $k, S, \bar{S}, C, \bar{C}$ defined by

$$\begin{aligned} &S := \sinh 2K, \qquad \bar{S} := \sinh 2\bar{K}, \qquad C := \cosh 2K, \\ &\bar{C} := \cosh 2\bar{K}, \qquad k := S\bar{S}. \end{aligned} \tag{7}$$

We will only treat the system in the ferromagnetic regime $K, \bar{K} > 0$ and $k \in (0, \infty)$, which exhibits a phase transition at the critical value $k = 1$. We will find subsequently that, from the point of view of the theory of isomonodromic systems, the next-to-diagonal correlations are functions of the two complex variables, k and one of $S, \bar{S}$, with k playing the role of the deformation variable and $-\bar{S}/S$ the spectral variable. While all of the results can be continued into the complex plane $k, S \in \mathbb{C}$ suitably restricted, we may often only state them for the physical regime $k, S, \bar{S} \in (0, \infty)$. Corresponding to the Ising model is a dual partner Ising model, which is related to the original by the duality transformation or involution

$$\sigma_r \mapsto \mu_r, \qquad \langle\sigma_{r_1}\ldots\sigma_{r_n}\rangle \mapsto \langle\mu_{r_1}\ldots\mu_{r_n}\rangle, \tag{8}$$

$$k \mapsto \frac{1}{k}, \qquad S \mapsto \frac{1}{\bar{S}}, \qquad \bar{S} \mapsto \frac{1}{S}. \tag{9}$$

The dynamic variables μ_r are known as the disorder variables and can be given an interpretation in terms of the spin variables σ_r [12].

The appearance of Toeplitz determinants such as those of (1) is indicative of several structures, and the most general of these is averages over the unitary group. Let $U \in U(N)$ have eigenvalues $z_1 = \mathrm{e}^{\mathrm{i}\theta_1}, \ldots, z_N = \mathrm{e}^{\mathrm{i}\theta_N}$. The unitary group $U(N)$ with Haar (uniform) measure has eigenvalue probability density function

$$\frac{1}{(2\pi)^N N!}\prod_{1\leqslant j<k\leqslant N}|z_k - z_j|^2, \qquad z_l := \mathrm{e}^{\mathrm{i}\theta_l} \in \mathbb{T}, \qquad \theta_l \in (-\pi, \pi], \tag{10}$$

with respect to the Lebesgue measure $\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N$ (see e.g. [5, chapter 2]), where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A well-known identity [18] relates averages of class functions, in particular products of a function $w(z)$ over the eigenvalues, to the Toeplitz determinant

$$I_N^{\epsilon}[w] := \left\langle \prod_{l=1}^{N} w(z_l) z_l^{\epsilon} \right\rangle_{U(N)} = \det[w_{-\epsilon+j-k}]_{j,k=1,\ldots,N}, \qquad \epsilon \in \mathbb{Z}, \qquad N \geqslant 1. \tag{11}$$

By convention, we set $I_0^\epsilon = 1$ and use the short-hand notation $I_N := I_N^0$. We identify $w(z)$ as a weight function with the Fourier decomposition

$$w(z) = \sum_{l\in\mathbb{Z}} w_l z^l. \tag{12}$$

The specific Fourier coefficients appearing in the diagonal Ising correlations (1) are

$$a_n(k) = \int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\zeta^n\sqrt{\frac{1-k^{-1}\zeta^{-1}}{1-k^{-1}\zeta}} = \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\frac{k\cos n\theta - \cos(n-1)\theta}{\sqrt{k^2+1-2k\cos\theta}}. \tag{13}$$

The implied weight is

$$a(\zeta;k) = \begin{cases} k^{-1/2}\zeta^{1/2}(\zeta-k^{-1})^{-1/2}(k-\zeta)^{1/2}, & 1<k<\infty \\ -k^{-1/2}\zeta^{1/2}(k^{-1}-\zeta)^{-1/2}(\zeta-k)^{1/2}, & 0\leqslant k<1. \end{cases} \tag{14}$$

The analytic structure is different depending on $k > 1$ (low temperature phase) or $k < 1$ (high temperature phase). The reason for the phase change of $\mathrm{e}^{-\pi\mathrm{i}}$ in the weight is because of the argument changes

$$\zeta - k = \mathrm{e}^{-\pi\mathrm{i}}(k-\zeta), \qquad k^{-1}-\zeta = \mathrm{e}^{\pi\mathrm{i}}(\zeta-k^{-1}), \tag{15}$$

as k goes from the $k > 1$ to the $k < 1$ regime. The correlation function for the disorder variables is

$$\langle\mu_{0,0}\mu_{N,N}\rangle = \det(\tilde{a}_{i-j}(k))_{1\leqslant i,j\leqslant N}, \tag{16}$$

where

$$\tilde{a}_n(k) = \int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\zeta^n\sqrt{\frac{1-k\zeta^{-1}}{1-k\zeta}} = \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{2\pi}\frac{\cos n\theta - k\cos(n-1)\theta}{\sqrt{k^2+1-2k\cos\theta}}. \tag{17}$$

The weight is

$$\tilde{a}(\zeta;k) = \begin{cases} -k^{1/2}\zeta^{1/2}(k-\zeta)^{-1/2}(\zeta-k^{-1})^{1/2}, & 1<k<\infty \\ k^{1/2}\zeta^{1/2}(\zeta-k)^{-1/2}(k^{-1}-\zeta)^{1/2}, & 0\leqslant k<1. \end{cases} \tag{18}$$

Although we use the same notation for the Toeplitz elements as Au-Yang and Perk [1], the relationship between our elements and theirs is $a_n = a_{-n}^{\mathrm{A-YP}}$ and $\tilde{a}_n = \tilde{a}_{-n}^{\mathrm{A-YP}}$.

From the viewpoint of the work [8] the weights (14), (18) are particular examples of the *regular semi-classical class*, characterized by a special structure of their logarithmic derivatives

$$\frac{1}{w(z)}\frac{\mathrm{d}}{\mathrm{d}z}w(z) := \frac{2V(z)}{W(z)} = \sum_{j=1}^{3}\frac{\rho_j}{z-z_j}, \qquad \rho_j\in\mathbb{C}. \tag{19}$$

Here $V(z)$, $W(z)$ are polynomials with $\deg V(z) < 3$, $\deg W(z) = 3$, respectively. The data for the weight (14) are then

$$\{z_j\}_{j=1}^3 = \{0, k^{-1}, k\}, \qquad \{\rho_j\}_{j=1}^3 = \left\{\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right\}. \tag{20}$$

The data for the other weight are (18)

$$\{z_j\}_{j=1}^3 = \{0, k, k^{-1}\}, \qquad \{\rho_j\}_{j=1}^3 = \left\{\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right\}, \tag{21}$$

which is the same as the previous case except for the exchange in the position of variable singularities. A particular observation in the Ising model case is that the Toeplitz matrix is not Hermitian, and the weight $w(z)$ is complex for real and physical $k\in(0,\infty)$, $z\in\mathbb{T}$. The duality transformation is simply a transposition of the singular points $z_2 \leftrightarrow z_3$ and at the critical temperature these two singularities coalesce.

An important identity relating the dual Toeplitz elements to the direct ones is the following well-known duality relation [1].

Proposition 1. *For all k and n, we have*

$$\tilde{a}_n(k) = a_n(k^{-1}) = -a_{-n+1}(k). \tag{22}$$

The two weights are related by the duality transformation

$$\tilde{a}(\zeta; k) = a(\zeta; k^{-1}). \tag{23}$$

By regarding the Fourier integral in (13) as a contour integral and changing the contour of integration, one obtains the well-known fact that the Toeplitz elements in the low temperature regime are given by

$$a_n = -\frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\pi\Gamma(n+1)} k^{-n}{}_2F_1\left(\frac{1}{2}, n-\frac{1}{2}; n+1; k^{-2}\right), \qquad n \geqslant 0, \tag{24}$$

$$a_{-n} = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\pi\Gamma(n+1)} k^{-n}{}_2F_1\left(-\frac{1}{2}, n+\frac{1}{2}; n+1; k^{-2}\right), \qquad n \geqslant 0, \tag{25}$$

whilst those in the high temperature regime are

$$a_n = -\frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\pi\Gamma(n)} k^{n-1}{}_2F_1\left(-\frac{1}{2}, n-\frac{1}{2}; n; k^2\right), \qquad n \geqslant 1. \tag{26}$$

$$a_{-n} = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\pi\Gamma(n+2)} k^{n+1}{}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; n+2; k^2\right), \qquad n \geqslant -1, \tag{27}$$

These elements are expressible as linear combinations of the complete first and second elliptic integrals K, E with arguments k^{-1} and k respectively [9] and with coefficient polynomials in these arguments. In the ensuing discussion, we adopt the following shorthand notation for the complete elliptic integrals of the first kind:

$$\mathrm{K}_< := \mathrm{K}(k), \qquad \mathrm{K}_> := \mathrm{K}(k^{-1}), \qquad \mathrm{K}_\diamondsuit := \mathrm{K}(k_\diamondsuit), \tag{28}$$

with analogous notation for the second kind and where $k_\diamondsuit = 2\sqrt{k}/(k+1)$ is the inverse Landen transformation. The complementary modulus is defined as $k'_\diamondsuit := \sqrt{1-k_\diamondsuit^2}$.

The complex weight $w(z)$ with support contained in $\mathbb{T}$ implicitly defines a system of *bi-orthogonal polynomials* $\{\phi_n(z), \bar{\phi}_n(z)\}_{n=0}^{\infty}$ on the unit circle by the orthogonality relation

$$\int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} w(\zeta)\phi_m(\zeta)\bar{\phi}_n(\bar{\zeta}) = \delta_{m,n}, \tag{29}$$

whose existence is ensured if and only if $I_n \neq 0$ for all $n \in \mathbb{N}$. Not withstanding the notation, $\bar{\phi}_n$ is not in general equal to the complex conjugate of ϕ_n and independent of it. The leading and trailing coefficients of these polynomials,

$$\phi_n(z) = \kappa_n z^n + \cdots + \phi_n(0), \qquad \bar{\phi}_n(z) = \kappa_n z^n + \cdots + \bar{\phi}_n(0),$$

occupy an important role in the theory where again $\bar{\phi}_n(0)$ are not in general equal to the corresponding complex conjugate. With the so-called reflection or Verblunsky coefficients specified by

$$r_n := \frac{\phi_n(0)}{\kappa_n}, \qquad \bar{r}_n := \frac{\bar{\phi}_n(0)}{\kappa_n}, \tag{30}$$

it is a well-known result in the theory of Toeplitz determinants that

$$\frac{I_{n+1}[w]I_{n-1}[w]}{(I_n[w])^2} = 1 - r_n\bar{r}_n, \qquad \kappa_n^2 = \frac{I_n}{I_{n+1}}, \quad n \geqslant 1. \tag{31}$$

Rather than dealing with $\bar{\phi}_n$, it is advantageous to define the reciprocal polynomial $\phi_n^*(z)$ by

$$\phi_n^*(z) := z^n\bar{\phi}_n(1/z). \tag{32}$$

In addition to the polynomial pair ϕ_n, ϕ_n^*, we require two non-polynomial solutions of the fundamental recurrence relations appearing in the theory [8],

$$\epsilon_n(z) := \int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{\zeta+z}{\zeta-z}w(\zeta)\phi_n(\zeta), \qquad n \geqslant 1, \tag{33}$$

$$\epsilon_n^*(z) := \frac{1}{\kappa_n} - \int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{\zeta+z}{\zeta-z}w(\zeta)\phi_n^*(\zeta), \qquad n \geqslant 1. \tag{34}$$

These form a matrix system

$$Y_n(z;t) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z)/w(z) \\ \phi_n^*(z) & -\epsilon_n^*(z)/w(z) \end{pmatrix}, \tag{35}$$

which, for regular semi-classical weights, has the property [8] that their monodromy data in the complex spectral z-plane are preserved under arbitrary deformations of the singularities z_j.

From the Toeplitz determinant formula (1), we observe that

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = \det[a_{j-k}]_{j,k=0,\ldots,N-1} = I_N[a(\zeta;k)] := I_N(k), \tag{36}$$

and apply the known results of subsection 3.1 in [7] which provides the following recurrence scheme for the diagonal correlations.

Corollary 1 [7]. *The diagonal correlation function for the Ising model valid in both the low and high temperature phases for $N \geqslant 1$ is determined by*

$$\frac{\langle\sigma_{0,0}\sigma_{N+1,N+1}\rangle\langle\sigma_{0,0}\sigma_{N-1,N-1}\rangle}{\langle\sigma_{0,0}\sigma_{N,N}\rangle^2} = 1 - r_N\bar{r}_N, \tag{37}$$

along with the quasi-linear $2/1$

$$\begin{aligned}(2N+3)(1-r_N\bar{r}_N)r_{N+1} &- 2N[k+k^{-1}+(2N-1)r_N\bar{r}_{N-1}]r_N \\ &+ (2N-3)[(2N-1)r_N\bar{r}_N+1]r_{N-1} = 0,\end{aligned} \tag{38}$$

and $1/2$ *recurrence relation*

$$\begin{aligned}(2N+1)(1-r_N\bar{r}_N)\bar{r}_{N+1} &- 2N[k+k^{-1}-(2N-3)\bar{r}_N r_{N-1}]\bar{r}_N \\ &+ (2N-1)[-(2N+1)r_N\bar{r}_N+1]\bar{r}_{N-1} = 0,\end{aligned} \tag{39}$$

subject to initial conditions $r_0 = \bar{r}_0 = 1$ *and*

$$r_1 = \begin{cases} \dfrac{k^2-2}{3k} + \dfrac{1-k^2}{3k}\dfrac{\mathrm{K}_>}{\mathrm{E}_>}, & 1 < k < \infty \\ \dfrac{1}{3}\left[-\dfrac{2}{k} + \dfrac{k\mathrm{E}_<}{(k^2-1)\mathrm{K}_< + \mathrm{E}_<}\right], & 0 \leqslant k < 1, \end{cases} \tag{40}$$

$$= \frac{1}{3}\left[-2\frac{1+k'_\diamond}{1-k'_\diamond} + \frac{1-k'_\diamond}{1+k'_\diamond}\frac{\mathrm{E}_\diamond + k'_\diamond\mathrm{K}_\diamond}{\mathrm{E}_\diamond - k'_\diamond\mathrm{K}_\diamond}\right], \tag{41}$$

$$\bar{r}_1 = \begin{cases} k + \dfrac{1-k^2}{k}\dfrac{\mathrm{K}_>}{\mathrm{E}_>}, & 1 < k < \infty \\ \dfrac{k\mathrm{E}_<}{(k^2-1)\mathrm{K}_< + \mathrm{E}_<}, & 0 \leqslant k < 1, \end{cases} \tag{42}$$

$$= \frac{1-k'_\diamondsuit}{1+k'_\diamondsuit}\frac{\mathrm{E}_\diamondsuit + k'_\diamondsuit \mathrm{K}_\diamondsuit}{\mathrm{E}_\diamondsuit - k'_\diamondsuit \mathrm{K}_\diamondsuit}. \tag{43}$$

The initial values of the correlations are

$$\langle \sigma_{0,0}\sigma_{1,1}\rangle = a_0 = \begin{cases} \dfrac{2}{\pi}\mathrm{E}_>, & 1 < k < \infty \\ \dfrac{2}{\pi k}[(k^2-1)\mathrm{K}_< + \mathrm{E}_<], & 0 \leqslant k < 1 \end{cases} \tag{44}$$

$$= \frac{2}{\pi}\frac{1}{1-k'_\diamondsuit}\left[\mathrm{E}_\diamondsuit - k'_\diamondsuit \mathrm{K}_\diamondsuit\right]. \tag{45}$$

A consequence of the duality relation (1) is the following obvious relations amongst the coefficients of the bi-orthogonal polynomial system.

Proposition 2. *For all n and k, we have*

$$I_n^{\varepsilon}[\tilde{a}] = (-1)^n I_n^{-1-\varepsilon}[a] \tag{46}$$

$$\bar{r}_n[\tilde{a}] = \frac{1}{\bar{r}_n[a]}. \tag{47}$$

Now we turn our attention to the object of the present study—the evaluation of the next-to-diagonal correlations. Let us recall that the elements b_n of the bordered Toeplitz determinant (4) can be written as

$$b_n = \bar{C}\int_{\mathbb{T}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\frac{\zeta^n}{\bar{S}+S\zeta}\sqrt{\frac{k/\zeta - 1}{k\zeta - 1}}. \tag{48}$$

These elements will also have complete elliptic function representations; however, for the anisotropic model we require the complete third elliptic integral defined by

$$\Pi(n,k) := \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}\frac{1}{1-n\sin^2\phi}. \tag{49}$$

We also adopt a notational shorthand for these, analogous to that for the first and second integrals

$$\Pi_< := \Pi(-S^2, k), \qquad \Pi_> := \Pi(-1/\bar{S}^2, k^{-1}), \qquad \Pi_\diamondsuit := \Pi(-4k(\bar{S}-S)^{-2}, k_\diamondsuit). \tag{50}$$

We note that $\Pi_\diamondsuit$ is not analytic at $\bar{S} = S$ and in fact has a discontinuity there of the following form:

$$\Pi_\diamondsuit = \frac{\pi}{2}\,\mathrm{sgn}(\bar{S}-S) + \mathrm{O}(\bar{S}-S) \quad \text{as} \quad \bar{S} \to S. \tag{51}$$

The first correlation in this sequence ($N = 1$) has the elliptic function evaluation

$$\langle\sigma_{0,0}\sigma_{1,0}\rangle = b_0 = \begin{cases} \dfrac{2\bar{C}}{\pi k S}[C^2\Pi_> - \mathrm{K}_>], & 1 < k < \infty \\ \dfrac{2\bar{C}}{\pi S}[C^2\Pi_< - \mathrm{K}_<], & 0 \leqslant k < 1, \end{cases} \tag{52}$$

$$= \frac{\bar{C}(1+k'_\diamond)}{2\pi S}\left[C^2\frac{\bar{S}+S}{\bar{S}-S}\Pi_\diamond + (S^2-1)\mathrm{K}_\diamond\right] + \frac{C}{S}\Theta(S-\bar{S}), \quad 0 \leqslant k < \infty, \tag{53}$$

where $\Theta(x)$ is the Heaviside step function. The term with the step function in (53) is necessary to compensate for the discontinuity in $\Pi_\diamond$, as given in (51), in order that the correlation function remains continuous at $\bar{S} = S$. The second correlation function ($N = 2$) has the evaluation

$$\langle\sigma_{0,0}\sigma_{2,1}\rangle = \begin{cases} \dfrac{4\bar{C}}{\pi^2 k^3 S}\{C^2[k^2(1-\bar{S}^2)\mathrm{E}_> + (k^2-1)\bar{S}^2\mathrm{K}_>]\Pi_> \\ \quad + k^4\mathrm{E}_>^2 + (1-k^2)\bar{S}^2\mathrm{K}_>^2 + k^2(\bar{S}^2-k^2)\mathrm{E}_>\mathrm{K}_>\}, & 1 < k < \infty \\ \dfrac{4\bar{C}}{\pi^2 k S}\{C^2[(k^2-1)\mathrm{K}_< + (1-\bar{S}^2)\mathrm{E}_<]\Pi_< \\ \quad + \mathrm{E}_<^2 + (1-k^2)\mathrm{K}_<^2 + (C^2\bar{S}^2-2)\mathrm{E}_<\mathrm{K}_<\}, & 0 \leqslant k < 1, \end{cases} \tag{54}$$

$$= \frac{\bar{C}}{\pi^2 S}\frac{1+k'_\diamond}{1-k'_\diamond}\left\{C^2[(1-\bar{S}^2)\mathrm{E}_\diamond - k'_\diamond\bar{C}^2\mathrm{K}_\diamond]\left(\frac{\bar{S}+S}{\bar{S}-S}\Pi_\diamond + \frac{2\pi}{1+k'_\diamond}\frac{\Theta(S-\bar{S})}{C\bar{C}}\right)\right.$$
$$\left. + \frac{4}{(1+k'_\diamond)^2}\mathrm{E}_\diamond^2 + k'_\diamond(\bar{S}^2-S^2)\mathrm{K}_\diamond^2 - (1-S^2)(1-\bar{S}^2)\mathrm{E}_\diamond\mathrm{K}_\diamond\right\}. \tag{55}$$

The correlation functions for the disorder variables or dual correlations are given by

$$\langle\mu_{0,0}\mu_{N,N-1}\rangle = \det\begin{pmatrix} \tilde{a}_0 & \cdots & \tilde{a}_{-N+2} & \tilde{b}_{N-1} \\ \tilde{a}_1 & \cdots & \tilde{a}_{-N+3} & \tilde{b}_{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{N-1} & \cdots & \tilde{a}_1 & \tilde{b}_0 \end{pmatrix}, \qquad N \geqslant 1, \tag{56}$$

where

$$\tilde{b}_n = C\bar{S}\int_{\mathbb{T}}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\frac{\zeta^{n-1}}{\bar{S}+S\zeta}\sqrt{\frac{1-k\zeta}{1-k/\zeta}}. \tag{57}$$

The correlations in this sequence also have elliptic function evaluations analogous to (52)–(55), but we refrain from writing these down as they can be obtained from the direct correlations using the duality transformation

$$\langle\mu_{0,0}\mu_{N,N-1}\rangle = \langle\sigma_{0,0}\sigma_{N,N-1}\rangle\Big|_{\substack{k\mapsto 1/k \\ S\mapsto 1/\bar{S} \\ \bar{S}\mapsto 1/S}}. \tag{58}$$

In addition, the $\langle\sigma_{0,0}\sigma_{N-1,N}\rangle$ correlations can be obtained from $\langle\sigma_{0,0}\sigma_{N,N-1}\rangle$ under the exchange $S \leftrightarrow \bar{S}$.

These correlation functions are in fact characterized as a solution to an *isomonodromic deformation* problem associated with the particular sixth Painlevé system, which itself characterizes the diagonal correlation functions. This observation is the key result of the present study.

Proposition 3. *The next-to-diagonal correlation functions are given by the second type of associated functions (34) appropriate to the weight (14) evaluated at a specific value of the spectral variable*

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \frac{\bar{C}}{2\bar{S}}\frac{I_{N-1}}{\kappa_{N-1}}\epsilon^*_{N-1}(z=-\bar{S}/S), \tag{59}$$

and valid for $N \geqslant 1$. *Here,* I_N *and* κ_N *are defined respectively by (11) and (31) appropriate to the weight (14).*

Proof. A result in the general theory of bi-orthogonal polynomials is the determinantal representation with a Toeplitz structure for the reciprocal polynomial [8]

$$\phi^*_n(z) = \frac{\kappa_n}{I^0_n}\det\begin{pmatrix} w_0 & \dots & w_{-n+1} & z^n \\ \vdots & \vdots & \vdots & \vdots \\ w_{n-j} & \dots & w_{-j+1} & z^j \\ \vdots & \vdots & \vdots & \vdots \\ w_n & \dots & w_1 & 1 \end{pmatrix}. \tag{60}$$

Using this and the definition of the second associated function (34), one obtains an analogous bordered Toeplitz determinant [19]

$$\epsilon^*_n(z) = \frac{\kappa_n}{I_n}\det\begin{pmatrix} w_0 & \dots & w_{-n+1} & g_n \\ \vdots & \vdots & \vdots & \vdots \\ w_{n-j} & \dots & w_{-j+1} & g_j \\ \vdots & \vdots & \vdots & \vdots \\ w_n & \dots & w_1 & g_0 \end{pmatrix}, \tag{61}$$

where

$$g_j(z) := -2z\int_{\mathbb{T}}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{\zeta^j}{\zeta-z}w(\zeta), \qquad z\notin\mathbb{T}. \tag{62}$$

The evaluation (59) then follows by a comparison of these last two formulae with (3) and (48), respectively. □

Many consequences flow from this identification; all of the general properties of the associated functions [8] can be applied. One particular useful characterization of the next-to-diagonal correlations is that they satisfy a linear three-term recurrence relation.

Corollary 2. *The associated function (59) satisfies the generic linear recurrence relation*

$$\frac{\kappa_n}{\kappa_{n+1}}\bar{r}_n\epsilon^*_{n+1}(z) + \frac{\kappa_{n-1}}{\kappa_n}\bar{r}_{n+1}z\epsilon^*_{n-1}(z) = [\bar{r}_n + \bar{r}_{n+1}z]\epsilon^*_n(z), \tag{63}$$

subject to the two initial values for $\epsilon^*_0, \epsilon^*_1$ *implied by (59) and (52)–(55). The auxiliary quantities appearing in (63) and (59) satisfy the generic recurrences*

$$I_{n+1} = \frac{I_n}{\kappa^2_n}, \qquad \kappa_{n+1} = \frac{\kappa_n}{\sqrt{1-r_{n+1}\bar{r}_{n+1}}}, \tag{64}$$

subject to their initial values

$$I_0 = 1, \qquad \kappa^2_0 = \frac{1}{a_0}, \tag{65}$$

utilizing (45).

We remark that this associated function also satisfies a linear second-order differential equation in the spectral variable z whose coefficients are determined by the auxiliary quantities discussed above. However, we refrain from writing this down as it does not appear to have as much practical utility as the recurrences in the above corollary.

To close our study we examine a number of limiting cases, namely the zero temperature, the critical temperature and high temperature limits. At zero temperature, $k \to \infty$, the solutions have leading order terms ($N \geqslant 1$)

$$r_N \underset{k\to\infty}{\sim} \frac{\left(-\frac{1}{2}\right)_N}{N!} k^{-N}, \qquad \bar{r}_N \underset{k\to\infty}{\sim} \frac{\left(\frac{1}{2}\right)_N}{N!} k^{-N} \qquad \langle \sigma_{0,0}\sigma_{N,N}\rangle \to 1. \tag{66}$$

At the critical point, $k = 1$, we have a complete solution for the bi-orthogonal system. The polynomial coefficients have the evaluations

$$\kappa_N^2 = \frac{\Gamma\left(N+\frac{3}{2}\right)\Gamma\left(N+\frac{1}{2}\right)}{\Gamma^2(N+1)}, \qquad r_N = -\frac{1}{(2N+1)(2N-1)}, \qquad \bar{r}_N = 1, \tag{67}$$

which is consistent with the well-known result [13]

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = \prod_{j=1}^{N} \frac{\Gamma^2(j)}{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(j-\frac{1}{2}\right)}. \tag{68}$$

The isomonodromic system is

$$\phi_N(z) = -\frac{\kappa_N}{(2N+1)(2N-1)} \cdot {}_2F_1\left(\frac{3}{2}, -N; -N+\frac{3}{2}; z\right), \tag{69}$$

$$\phi_N^*(z) = \kappa_N \cdot {}_2F_1\left(\frac{1}{2}, -N; -N+\frac{1}{2}; z\right), \tag{70}$$

$$\frac{1}{2}\kappa_N\epsilon_N(z) = -\frac{1}{(2N+3)(2N+1)z} \cdot {}_2F_1\left(\frac{3}{2}, N+1; N+\frac{5}{2}; 1/z\right), \tag{71}$$

$$\frac{1}{2}\kappa_N\epsilon_N^*(z) = {}_2F_1\left(\frac{1}{2}, N+1; N+\frac{3}{2}; 1/z\right). \tag{72}$$

The last result (72) is consistent with the critical next-to-diagonal correlation given in [2]

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \langle \sigma_{0,0}\sigma_{N,N}\rangle C \cdot {}_2F_1\left(\tfrac{1}{2}, N; N+\tfrac{1}{2}; -S^2\right). \tag{73}$$

At infinite temperature, $k \to 0$, the leading order terms are ($N \geqslant 1$)

$$r_N \underset{k\to 0}{\sim} \frac{\left(-\frac{1}{2}\right)_N}{(N+1)!} k^{-N}, \qquad \bar{r}_N \underset{k\to 0}{\sim} \frac{N!}{\left(\frac{1}{2}\right)_N} k^{N}, \qquad \langle \sigma_{0,0}\sigma_{N,N}\rangle \to 0, \tag{74}$$

and the series expansion of these about $k = 0$ in terms of the generalized hypergeometric function is given in [7].

Acknowledgments

This research has been supported by the Australian Research Council. The author would like to express his sincere gratitude for the generous assistance and guidance provided by Jacques Perk. He has also benefited from extensive discussions on all matters relating to the Ising model in its various aspects with J-M Maillard, B McCoy, T Miwa and J Palmer.

References

[1] Au-Yang H, Jin B-Q and Perk J H H 2001 Wavevector-dependent susceptibility in quasiperiodic Ising models *Proc. Baxter Revolution in Mathematical Physics (Canberra, 2000)* vol 102 pp 501–43

[2] Au-Yang H and Perk J H H 1987 Critical correlations in a Z-invariant inhomogeneous Ising model *Physica* A **144** 44–104

[3] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[4] Borodin A 2003 Discrete gap probabilities and discrete Painlevé equations *Duke Math. J.* **117** 489–542

[5] Forrester P J 2005 *Log Gases and Random Matrices,* http://www.ms.unimelb.edu.au/~matpjf/matpjf.html

[6] Forrester P J and Witte N S 2004 Application of the τ-function theory of Painlevé equations to random matrices: $\mathrm{P_{VI}}$, the JUE, CyUE, cJUE and scaled limits *Nagoya Math. J.* **174** 29–114

[7] Forrester P J and Witte N S 2005 Discrete Painlevé equations for a class of $P_{\mathrm{VI}}\,\tau$-functions given as $\mathrm{U}(N)$ averages *Nonlinearity* **18** 2061–88

[8] Forrester P J and Witte N S 2006 Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems *Construct. Approx.* **24** 201–37

[9] Ghosh R K and Shrock R E 1984 Exact expressions for diagonal correlation functions in the $d = 2$ Ising model *Phys. Rev.* B **30** 3790–4

[10] Jimbo M and Miwa T 1980 Studies on holonomic quantum fields: XVII *Proc. Japan Acad. Ser. A Math. Sci.* **56** 405–10

[11] Jimbo M and Miwa T 1981 *Proc. Japan Acad. Ser. A Math. Sci.* **57** 347 (erratum)

[12] Kadanoff L P and Kohmoto M 1980 SMJ's analysis of Ising model correlation functions *Ann. Phys.* **126** 371–98

[13] McCoy B and Wu T T 1973 *The Two-Dimensional Ising Model* (Harvard: Harvard University Press)

[14] Miwa T and Jimbo M 1982 *Introduction to Holonomic Quantum Fields* The Riemann problem, complete integrability and arithmetic applications *(Lecture Notes in Math.* vol 925) (Berlin: Springer) pp 28–36

[15] Okamoto K 1987 Studies on the Painlevé equations: I. Sixth Painlevé equation P_{VI} *Ann. Mat. Pura Appl.* **146** 337–81

[16] Palmer J 2007 Planar Ising correlations and the deformation analysis of scaling *Prog. Math. Phys.* vol 49 (Boston: Birkhäuser)

[17] Sato M, Miwa T and Jimbo M 1980 Aspects of holonomic quantum fields. isomonodromic deformation and Ising model *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory (Lecture Notes in Phys.* vol 126*)* (Berlin: Springer) pp 429–91

[18] Szegö G 1967 *Orthogonal Polynomials (Colloquium Publications* vol 23*)* 3rd edn (Providence, RI: American Mathematical Society)

[19] Witte N S 2007 Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems: II. (unpublished)